

Gravitational radiation reaction in compact binary systems: Contribution of the magnetic dipole-magnetic dipole interaction

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We study the gravitational radiation reaction in compact binary systems composed of neutron stars with spin and huge magnetic dipole moments (magnetars). The magnetic dipole moments undergo a precessional motion about the respective spins. At sufficiently high values of the magnetic dipole moments, their interaction generates second post-Newtonian order contributions both to the equations of motion and to the gravitational radiation escaping the system. We parametrize the radial motion and average over a radial period in order to find the secular contributions to the energy and magnitude of the orbital angular momentum losses, in the generic case of *eccentric* orbits. Similarly as for the spin-orbit, spin-spin, quadrupole-monopole interactions, here too we deduce the secular evolution of the relative orientations of the orbital angular momentum and spins. These equations, supplemented by the evolution equations for the angles characterizing the orientation of the dipole moments form a first order differential system, which is closed. The circular orbit limit of the energy loss agrees with Ioka and Taniguchi's earlier result.

I. INTRODUCTION

Neutron star and black hole binary systems are among the most probable sources of the gravitational radiation emitted in the frequency range of the Earth-based interferometric detectors such as the Laser Interferometric Gravitational Wave Observatory (LIGO) [1], VIRGO [2], GEO [3], and TAMA [4] (all will detect in the High-Frequency Band: 1 Hz to 10^4 Hz) and the envisaged Laser Interferometer Space Antenna (LISA)[5, 6]. (Low-Frequency Band: 10^{-4} Hz to 1 Hz) respectively. A post-Newtonian (PN) treatment of 3.5 PN orders is generally agreed to describe with sufficient accuracy both the motion and the gravitational radiation up to the point of the innermost circular orbit (ICO), at least in the case of neutron star binaries [7]. While the 3PN order approximation is expected to locate the ICO with an accuracy of 1% for binary systems with comparable masses, it breaks down in the region inner to the ICO in the latest stage of the final coalescence, when numerical relativity is required. However, there is a gap between the failure of the PN expansion and the beginning of the merger, this being called the intermediate binary black hole (IBBH) problem [8], recently assessed in [9].

The effort of providing templates for the search of the gravitational waves in the noisy background is underway, at least for binaries on circular orbits, characterized by their mass and the radius of the orbit. The assumption of circular orbits is justified by the circularizing property of the gravitational waves [10], still interesting situations can occur in galactic nuclei, when freshly formed binary systems have not enough time to circularize [11, 12]. The error produced by the neglect of the eccentricity during the search for gravitational waves was estimated to be substantial [13]. The assumption of disregarding other characteristics of the binary components, like the spins, quadrupole moments and even magnetic dipole moments (to be discussed in the present paper), in many cases is not well founded. For example there is increasing observational evidence for an overwhelming majority of galactic black holes (even colliding ones) with maximally allowed spins [14]. While neutron star binaries are not expected to be significantly spinning [15], [16], neutron stars with super-strong magnetic fields (magnetars) have also been detected [17].

Therefore a generic treatment, allowing eccentric orbits and all types of contributions up to the 3.5thPN order would be desirable in the PN-description of gravitational waves. The complexity of such results forbids their immediate implementation in templates, however with increasing computational capacity this is hoped to become possible. An other complicating feature arising in the eccentric case is the occurrence of several harmonics in the incoming gravitational wave which, contrarily to the circular case (where the dominant frequency is twice of the orbital frequency) will have comparable intensities [18]. In the search for precessing binaries the use of effective templates has been proposed recently [19], [20].

The eccentricity was taken into account in the second PN order description of [21]. The 1.5 PN spin-orbit and 2PN spin-spin contributions to the orbital evolution and gravitational radiation reaction were considered in [22, 23, 24, 25, 26, 27, 28, 29]. The gravitational quadrupole-monopole type radiative contribution was also treated recently [30], the results matching the previous circular orbit computation [31].

The effect of possible strong magnetic fields of the compact binary components was discussed by Ioka and Taniguchi [32], for the case of circular orbits. They argue that in the upper limit of 10^{16} G for the magnetic fields the coupling

of the magnetic dipole moments generates contributions of the same magnitude than the 2PN corrections to both the equation of motion and the gravitational radiation. They also compute the corresponding electromagnetic radiation, and find out that the electromagnetically radiated power is "much less" than the power radiated away gravitationally. The ratio of the two types of radiations is 3/64 cf. Eq. (16) of [32], which counts approximately half of a PN order. Being interested in the *leading order radiation due to the magnetic dipoles*, we will disregard the electromagnetic component of radiation. Whether the leading order gravitational radiation due to magnetic moments truly appears at second or only at higher orders depends on the strength of the magnetic fields. Actual observational evidence supports magnetic fields of 10^{15} G for isolated magnetars [17].

In the present paper we would like to extend our previous computations carried on for the spin-orbit (SO), spin-spin (SS) and quadrupole-monopole (QM) interactions for the case of the magnetic dipole - magnetic dipole interaction. This would mean a generalization of Ioka and Taniguchi's description allowing for eccentric orbits.

In Sect. II we set up the formalism, by introducing the generalized true and eccentric anomaly parametrizations for the magnetic dipole - magnetic dipole perturbation. As proved earlier [33], all relevant integral expressions can be easily computed by use of the residue theorem, with the additional bonus that in the majority of cases the only pole is at the origin. Our treatment follows closely Refs. [28, 29] and [30]. In contrast with the energy E , the magnitude of the orbital angular momentum is not a conserved quantity in the absence of radiation at this order. Thus we introduce its angular average \bar{L} in order to characterize the perturbed radial motion. The radial period, as well as the relation between \bar{L} and the time average $\langle L \rangle$ resemble the results of the previously discussed cases.

Sect. III contains the main results of the paper. These are the secular evolutions of E and \bar{L} due to the magnetic dipole - magnetic dipole contribution to the gravitational radiation. Also, as in previous cases, the evolution of the angle variables κ_i and γ , which characterize the relative orientation of the spin vectors and orbital angular momentum vector are derived. However the evolution of this set of variables $(E, \bar{L}, \kappa_i, \gamma)$ does not close to a first order differential system, as in previous cases. In order to close the system, we have to compute the evolution equations for the angle variables β_i characterizing the relative orientation of the spins and magnetic dipole moments. The reliability of our results is checked in the Concluding Remarks, where in the circular limit we recover the expression for the energy loss given in [32].

The velocity of light c and the gravitational constant G are kept in all expressions.

II. THE RADIAL MOTION

We consider a binary system composed of neutron stars with magnetic dipole moments \mathbf{d}_i (properly scaled in order to absorb all dimension-carrying constants). The dipole-dipole interaction is given by the Lagrangian [32]:

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_{DD} = \frac{\mu}{2} \mathbf{v}^2 + \frac{Gm\mu}{r} + \frac{1}{r^3} [3(\mathbf{n} \cdot \mathbf{d}_1)(\mathbf{n} \cdot \mathbf{d}_2) - \mathbf{d}_1 \cdot \mathbf{d}_2] , \quad (1)$$

where $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{r} = r\mathbf{n}$. Accordingly, the total acceleration is the sum of the Newtonian \mathbf{a}_N and the dipole-dipole contribution \mathbf{a}_{DD} :

$$\begin{aligned} \mathbf{a} &= \mathbf{a}_N + \mathbf{a}_{DD} , \\ \mathbf{a}_N &= -\frac{Gm}{r^2} \mathbf{n} , \\ \mathbf{a}_{DD} &= \frac{3}{\mu r^4} \left\{ [\mathbf{d}_1 \cdot \mathbf{d}_2 - 5(\mathbf{n} \cdot \mathbf{d}_1)(\mathbf{n} \cdot \mathbf{d}_2)] \mathbf{n} + \sum_{i \neq j} (\mathbf{n} \cdot \mathbf{d}_i) \mathbf{d}_j \right\} . \end{aligned} \quad (2)$$

The orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{L}_N$ evolves due to the dipole-dipole perturbation as

$$\dot{\mathbf{L}} = \frac{3}{r^3} \sum_{i \neq j} (\mathbf{n} \cdot \mathbf{d}_i) (\mathbf{n} \times \mathbf{d}_j) , \quad (3)$$

and its magnitude L changes accordingly

$$\dot{L} = \dot{\mathbf{L}} \hat{\mathbf{L}} = \frac{3\mu}{Lr^2} \sum_{i \neq j} (\mathbf{n} \cdot \mathbf{d}_i) (\mathbf{v} - \dot{r}\mathbf{n}) \cdot \mathbf{d}_j . \quad (4)$$

The energy $E = \dot{\mathbf{r}} \partial \mathcal{L} / \partial \dot{\mathbf{r}} - \mathcal{L}$ and the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2$ are the constants of this motion.

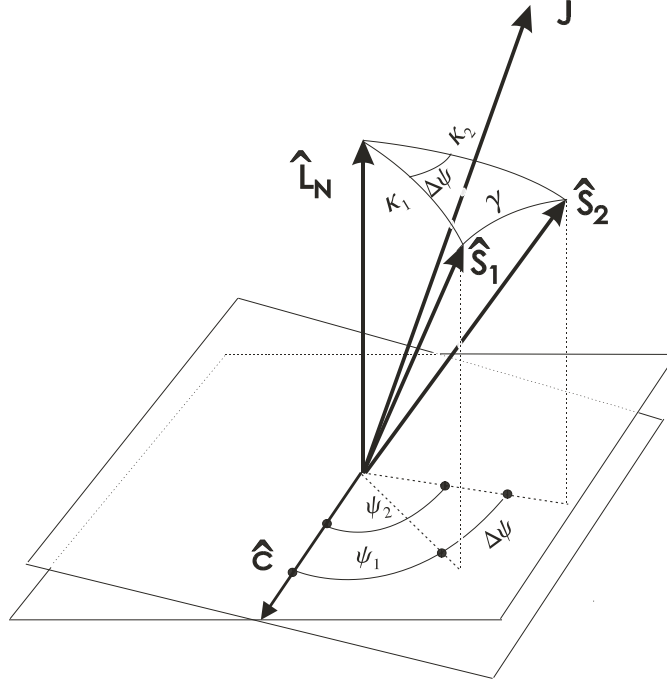


FIG. 1: The relative angles κ_i and γ of the \mathbf{L}_N and \mathbf{S}_i angular momenta and the azimuthal angles ψ_i of the spins measured in the plane perpendicular to \mathbf{L} . The vector $\hat{\mathbf{c}}$ lies on the intersection of this plane with the one perpendicular to \mathbf{J} .

We introduce three orthonormal coordinate systems \mathcal{K} and \mathcal{K}^i with the axes $(\hat{\mathbf{c}}, \hat{\mathbf{L}} \times \hat{\mathbf{c}}, \hat{\mathbf{L}})$ and $(\hat{\mathbf{b}}_i, \hat{\mathbf{S}}_i \times \hat{\mathbf{b}}_i, \hat{\mathbf{S}}_i)$, where $\hat{\mathbf{c}}$ and $\hat{\mathbf{b}}_i$ are the unit vectors in the $\mathbf{J} \times \mathbf{L}$ and $\mathbf{S}_i \times \mathbf{L}$ directions, respectively. In the system \mathcal{K} the polar angles κ_i and ψ_i of the spins are defined as $\hat{\mathbf{S}}_i = (\sin \kappa_i \cos \psi_i, \sin \kappa_i \sin \psi_i, \cos \kappa_i)$ (Fig 1). We also introduce the polar angles α_i and β_i of the the magnetic dipole moments \mathbf{d}_i in the corresponding system \mathcal{K}^i as $\hat{\mathbf{d}}_i = (\sin \alpha_i \cos \beta_i, \sin \alpha_i \sin \beta_i, \cos \alpha_i)$ (Fig 2). The transformation $\mathcal{K}^i \rightarrow \mathcal{K}$ represents a sequence of rotations $R_z(-\tau_i)R_x(-\kappa_i)$, where the angles $\tau_i = \cos^{-1}(\hat{\mathbf{c}} \cdot \hat{\mathbf{b}}_i)$ satisfy the relations $\tau_i + \psi_i = \pi/2$ (see Fig. 2 for a proof). Thus in the \mathcal{K} system the vectors appearing in Eq. (4) become¹:

$$\mathbf{n} = \begin{pmatrix} \cos \psi \\ \sin \psi \\ 0 \end{pmatrix}, \quad \mathbf{v} = \dot{r}\mathbf{n} + \frac{L}{\mu r} \begin{pmatrix} -\sin \psi \\ \cos \psi \\ 0 \end{pmatrix},$$

$$\mathbf{d}_i = d_i \begin{pmatrix} \rho_i \sin \psi_i + \sigma_i \cos \psi_i \\ \sigma_i \sin \psi_i - \rho_i \cos \psi_i \\ \zeta_i \end{pmatrix}, \quad (5)$$

where we have introduced the shorthand notations

$$\begin{aligned} \rho_i &= \sin \alpha_i \cos \beta_i, \\ \nu_i &= \sin \alpha_i \sin \beta_i, \\ \sigma_i &= \cos \alpha_i \sin \kappa_i + \nu_i \cos \kappa_i, \\ \zeta_i &= \cos \alpha_i \cos \kappa_i - \nu_i \sin \kappa_i. \end{aligned} \quad (6)$$

Since \mathcal{L}_{DD} is independent of the velocity, $E_{DD} = -\mathcal{L}_{DD}$ holds:

$$E_{DD} = \frac{d_1 d_2}{2r^3} [\mathcal{A}_0 - 3\mathcal{B}_2(\chi)]. \quad (7)$$

¹ Due to the magnetic dipole - magnetic dipole perturbation all three of the above defined coordinate systems may evolve (to DD -order). We do not consider these evolutions, as the components of the above vectors enter solely in various DD -terms, thus they are needed with Newtonian accuracy.

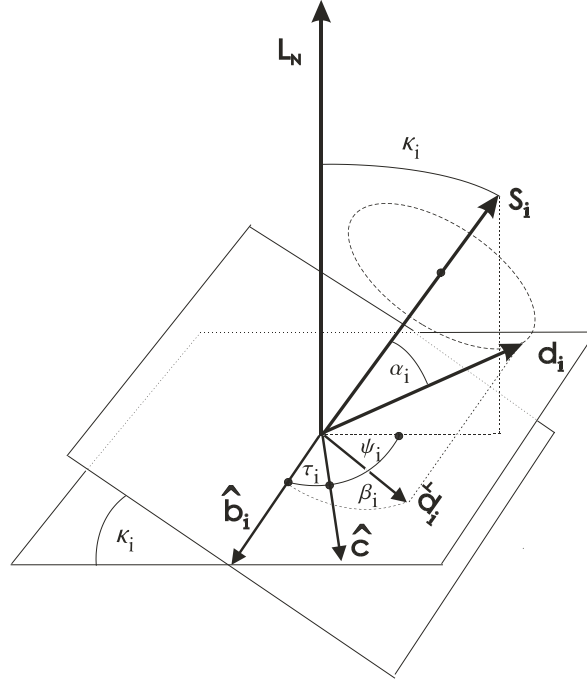


FIG. 2: The magnetic dipole moment of each neutron star undergoes a precessional motion about the spin vector of the respective neutron star. The projection of the magnetic dipole moment into the plane perpendicular to \mathbf{S}_i is denoted ${}^\perp\mathbf{d}_i$, a vector characterized by the azimuthal angle β_i . On the intersection of the planes perpendicular to \mathbf{L}_N and \mathbf{S}_i lies the vector $\hat{\mathbf{b}}_i$. As the projection of \mathbf{S}_i into the plane of motion lies in the plane defined by \mathbf{S}_i and \mathbf{L}_N , a plane to which the vector $\hat{\mathbf{b}}_i$ is perpendicular by definition, the relation $\tau_i + \psi_i = \pi/2$ holds.

Here we have introduced the Newtonian true anomaly parameter $\chi = \psi - \psi_0$ and

$$\mathcal{A}_0 = 2 \cos \lambda + 3(\rho_1 \sigma_2 - \rho_2 \sigma_1) \sin(\delta_1 - \delta_2) - 3(\rho_1 \rho_2 + \sigma_1 \sigma_2) \cos(\delta_1 - \delta_2) , \quad (8)$$

$$\mathcal{B}_k(\chi) = (\sigma_1 \sigma_2 - \rho_1 \rho_2) \cos(k\chi + \delta_1 + \delta_2) - (\rho_1 \sigma_2 + \rho_2 \sigma_1) \sin(k\chi + \delta_1 + \delta_2) , \quad (9)$$

where λ is the angle subtended by the magnetic moments and $\delta_i = \psi_0 - \psi_i$. We stress here that the angular average of (9) vanishes $\langle \mathcal{B}_k(\chi) \rangle = 0$ for any integer k .

With these notations, the evolution of the magnitude of angular orbital momentum is:

$$\dot{L} = \frac{3d_1 d_2}{2r^3} \mathcal{B}_2'(\chi) , \quad (10)$$

where a prime denotes derivative with respect to χ . In all DD expressions ρ_i and σ_i can be regarded as constants, since the evolution of the angles β_i and κ_i are of 1.5PN order and α_i are constants. Similar considerations hold for λ . After the integration $\int \dot{L} \dot{\chi}^{-1} d\chi$, we obtain

$$L(\chi) = L_0 - \frac{\mu^2 d_1 d_2}{2\bar{L}^3} \{ (3Gm\mu + 4\bar{A}) \mathcal{B}_0 - (3Gm\mu + 4\bar{A} \cos \chi) \mathcal{B}_2(\chi) + \bar{A} \sin \chi \mathcal{B}_2'(\chi) \} , \quad (11)$$

where \bar{A} is the magnitude of the Laplace-Runge-Lenz vector for a Keplerian motion characterized by E and \bar{L}

$$\bar{A} = \left(G^2 m^2 \mu^2 + \frac{2E\bar{L}^2}{\mu} \right)^{1/2} . \quad (12)$$

From (11) we see that $L(0) = L(2\pi) = L_0$ holds. Let \bar{L} denote the angular average of $L(\chi)$

$$\bar{L} = L_0 - \frac{\mu^2 d_1 d_2}{2\bar{L}^3} (3Gm\mu + 4\bar{A}) \mathcal{B}_0 . \quad (13)$$

Then we can express $L(\chi)$ in terms of \bar{L} rather than L_0 :

$$L(\chi) = \bar{L} + \delta L_{DD} , \quad (14)$$

$$\delta L_{DD} = \frac{\mu^2 d_1 d_2}{2\bar{L}^3} [(3Gm\mu + 4\bar{A} \cos \chi) \mathcal{B}_2(\chi) - \bar{A} \sin \chi \mathcal{B}_2'(\chi)] . \quad (15)$$

The magnitude of the velocity and the radial equation take the form

$$v^2 = \frac{2}{\mu} [E - E_{DD}(r, \chi)] + \frac{2Gm}{r} , \quad (16)$$

$$\dot{r}^2 = \frac{2}{\mu} [E - E_{DD}(r, \chi)] + \frac{2Gm}{r} - \frac{L(\chi)^2}{\mu^2 r^2} . \quad (17)$$

Turning points are defined as solutions of the $\dot{r} = 0$ equation. They can be expressed as

$$r_{\min}^{\max} = r_{\pm} \mp \frac{\mu r_{\pm}^2 E_{DD}^{\pm} + \bar{L} \delta L_{DD}^{\pm}}{\mu \bar{A}} , \quad (18)$$

where $E_{DD}^{\pm} = E_{DD}(r_{\pm})$, $\delta L_{DD}^{\pm} = \delta L_{DD}(r_{\pm})$ and r_{\pm} are the turning points for a Keplerian orbit characterized by E and \bar{A} . Substituting (7) and (15) we obtain

$$r_{\min}^{\max} = \frac{Gm\mu \pm \bar{A}}{-2E} + \frac{\mu d_1 d_2}{2\bar{A}\bar{L}^2} \{ (\bar{A} \mp Gm\mu) \mathcal{A}_0 + \bar{A} \mathcal{B}_0 \} . \quad (19)$$

Next we introduce the generalized true and eccentric anomaly parametrizations of the radial motion, following the generic recipe from Ref. [33]:

$$r = \frac{\bar{L}^2}{\mu(Gm\mu + \bar{A} \cos \chi)} + \frac{\mu d_1 d_2 \Lambda}{2\bar{A}\bar{L}^2 (Gm\mu + \bar{A} \cos \chi)^2} , \quad (20)$$

$$\Lambda = \bar{A} [(3G^2 m^2 \mu^2 + \bar{A}^2) \mathcal{A}_0 + (G^2 m^2 \mu^2 + \bar{A}^2) \mathcal{B}_0] + Gm\mu [(G^2 m^2 \mu^2 + 3\bar{A}^2) \mathcal{A}_0 + 2\bar{A}^2 \mathcal{B}_0] \cos \chi , \quad (21)$$

$$r = \frac{Gm\mu - \bar{A} \cos \xi}{-2E} + \frac{\mu d_1 d_2}{2\bar{A}\bar{L}^2} \{ \bar{A} (\mathcal{A}_0 + \mathcal{B}_0) + Gm\mu \mathcal{A}_0 \cos \xi \} . \quad (22)$$

Introducing suitable complex variables, radial expressions can be averaged by use of the residue theorem, as described in [33]. Following this recipe, the radial period turns out to have the Keplerian form

$$T = 2\pi Gm \left(\frac{\mu}{-2E} \right)^{3/2} . \quad (23)$$

(This holds whenever the radial dependence of the perturbing terms in the radial equation (17) is either $1/r^2$ or $1/r^3$ [33].) Time average of $L(\chi)$ gives

$$\langle L \rangle = \bar{L} + \frac{d_1 d_2 \mathcal{B}_0}{2\mu \bar{A}^2 \bar{L}^3} \left[Gm\mu^4 (2G^2 m^2 \mu^2 - 3\bar{A}^2) - 2(-2\mu E)^{3/2} \bar{L}^3 \right] . \quad (24)$$

This latter expression allows for an equivalent expression of all forthcoming results in terms of $\langle L \rangle$ rather than \bar{L} , case needed. We remark that the square bracket in Eq. (24) coincides with the factor F_1/F_2 of the corresponding expressions in the quadrupole-monopole and spin-spin cases, Refs. [30] and [28].

III. LEADING ORDER MAGNETIC DIPOLE-MAGNETIC DIPOLE CONTRIBUTION TO THE EVOLUTIONS OF THE DYNAMICAL VARIABLES UNDER RADIATION REACTION

A. Energy loss

The radiative change in the energy to leading order is given by the quadrupole formula

$$\frac{dE}{dt} = -\frac{G}{5c^5} I^{(3)jl} I^{(3)jl} , \quad (25)$$

where $I^{(3)jl}$ is the 3^{rd} time derivative of the system's symmetric trace-free (STF) mass quadrupole moment tensor. To leading order it is given as

$$I_N^{jl} = \mu (x^j x^l)^{STF} . \quad (26)$$

Inserting v^2 and \dot{r}^2 from (16) and (17), together with E_{DD} and $L(\chi)$ from (7) and (14) into the quadrupole formula we obtain

$$\frac{dE}{dt} = \left(\frac{dE}{dt} \right)_N + \left(\frac{dE}{dt} \right)_{DD} , \quad (27)$$

$$\left(\frac{dE}{dt} \right)_N = -\frac{8G^3 m^2}{15c^5 r^6} (2E\mu r^2 + 2Gm\mu^2 r + 11\bar{L}^2) , \quad (28)$$

$$\left(\frac{dE}{dt} \right)_{DD} = \frac{4G^2 m d_1 d_2}{15c^5 \mu \bar{L}^2 r^8} \left\{ \sum_{k=1}^3 a_k \mathcal{B}_k(\chi) + a_4 \mathcal{A}_0 \right\} \quad (29)$$

with the coefficients a_k given by

$$\begin{aligned} a_1 &= 3\mu \bar{A} r (-22Gm\mu^2 r + 17\bar{L}^2) , \\ a_2 &= 6(-11G^2 m^2 \mu^4 r^2 + 6E\bar{L}^2 \mu r^2 + 5Gm\mu^2 \bar{L}^2 r - 51\bar{L}^4) , \\ a_3 &= -\mu \bar{A} r (22Gm\mu^2 r + 51\bar{L}^2) , \\ a_4 &= 2\bar{L}^2 (-6E\mu r^2 - 5Gm\mu^2 r + 39\bar{L}^2) . \end{aligned} \quad (30)$$

To compute the averaged loss of the energy we parametrize the above expression by the true anomaly χ and pass to the complex variable $z = \exp(i\chi)$. The only pole of the integrand is located in the origin. Using the residue theorem we obtain

$$\left\langle \frac{dE}{dt} \right\rangle = \left\langle \frac{dE}{dt} \right\rangle_N + \left\langle \frac{dE}{dt} \right\rangle_{DD} , \quad (31)$$

$$\left\langle \frac{dE}{dt} \right\rangle_N = -\frac{G^2 m (-2E\mu)^{3/2}}{15c^5 \bar{L}^7} (148E^2 \bar{L}^4 + 732G^2 m^2 \mu^3 E \bar{L}^2 + 425G^4 m^4 \mu^6) , \quad (32)$$

$$\left\langle \frac{dE}{dt} \right\rangle_{DD} = \frac{G d_1 d_2 (-2E\mu)^{3/2}}{15c^5 \bar{L}^{11}} (C_1 \mathcal{B}_0 + C_2 \mathcal{A}_0) \quad (33)$$

where the coefficients $C_{1,2}$ are

$$\begin{aligned} C_1 &= -\mu \bar{A}^2 (948E^2 \bar{L}^4 + 8936G^2 E \bar{L}^2 m^2 \mu^3 + 8335G^4 m^4 \mu^6) , \\ C_2 &= 708E^3 \bar{L}^6 + 10020G^2 E \bar{L}^4 m^2 \mu^3 + 18865G^4 E \bar{L}^2 m^4 \mu^6 + 8316G^6 m^6 \mu^9 . \end{aligned} \quad (34)$$

B. Change in the magnitude of orbital momentum

Since there is no secular spin evolution in the 2PN order, the loss in the magnitude L under radiation reaction can be written as

$$\frac{dL}{dt} \simeq \hat{\mathbf{L}} \cdot \frac{d\mathbf{J}}{dt} , \quad (35)$$

where \simeq denotes equality modulo spin terms, which average out due to $\langle d\mathbf{S}_i/dt \rangle = 0$, see [29]. The instantaneous loss of the total angular momentum \mathbf{J} to leading order is

$$\frac{d\mathbf{J}^i}{dt} = -\frac{2G}{5c^5} \epsilon^{ijk} I^{(2)jl} I^{(3)kl} . \quad (36)$$

Inserting v^2 , \dot{r}^2 and the respective components of \mathbf{n} and \mathbf{d}_i from (5) into (35) we obtain

$$\hat{\mathbf{L}} \cdot \frac{d\mathbf{J}}{dt} = \left(\hat{\mathbf{L}} \cdot \frac{d\mathbf{J}}{dt} \right)_N + \left(\hat{\mathbf{L}} \cdot \frac{d\mathbf{J}}{dt} \right)_{DD} , \quad (37a)$$

$$\left(\hat{\mathbf{L}} \cdot \frac{d\mathbf{J}}{dt} \right)_N = \frac{8G^2 m \bar{L}}{5c^5 \mu r^5} (2E\mu r^2 - 3\bar{L}^2) , \quad (37b)$$

$$\left(\hat{\mathbf{L}} \cdot \frac{d\mathbf{J}}{dt} \right)_{DD} = \frac{4G d_1 d_2}{5c^5 \mu^2 \bar{L}^3 r^7} \left\{ \sum_{k=1}^3 b_k \mathcal{B}_k(\chi) + b_4 \mathcal{A}_0 \right\} , \quad (37c)$$

with the coefficients b_k given by

$$\begin{aligned} b_1 &= 3\mu\bar{A}r(2GEm\mu^3r^3 - E\bar{L}^2\mu r^2 - 9G\bar{L}^2m\mu^2r + 8\bar{L}^4) , \\ b_2 &= 3(2G^2Em^2\mu^5r^4 + 22E\bar{L}^4\mu r^2 - 9G^2\bar{L}^2m^2\mu^4r^2 + 10G\bar{L}^4m\mu^2r - 19\bar{L}^6) , \\ b_3 &= \mu\bar{A}r(2GEm\mu^3r^3 + 3E\bar{L}^2\mu r^2 - 9G\bar{L}^2m\mu^2r - 24\bar{L}^4) , \\ b_4 &= \bar{L}^4(-18E\mu r^2 - 8Gm\mu^2r + 15\bar{L}^2) . \end{aligned} \quad (38)$$

After averaging, we obtain the secular loss in the magnitude of orbital angular momentum:

$$\left\langle \frac{dL}{dt} \right\rangle = \left\langle \frac{dL}{dt} \right\rangle_N + \left\langle \frac{dL}{dt} \right\rangle_{DD} , \quad (39)$$

$$\left\langle \frac{dL}{dt} \right\rangle_N = -\frac{4G^2m(-2E\mu)^{3/2}}{5c^5\bar{L}^4} (14E\bar{L}^2 + 15G^2m^2\mu^3) , \quad (40)$$

$$\left\langle \frac{dL}{dt} \right\rangle_{DD} = \frac{Gd_1d_2(-2E\mu)^{3/2}}{5c^5\bar{L}^8} [D_1\mathcal{B}_0 + D_2\mathcal{A}_0] , \quad (41)$$

where the coefficients $D_{1,2}$ are given as

$$\begin{aligned} D_1 &= -6\mu\bar{A}^2(31E\bar{L}^2 + 90G^2m^2\mu^3) , \\ D_2 &= 252E^2\bar{L}^4 + 1200G^2E\bar{L}^2m^2\mu^3 + 805G^4m^4\mu^6 . \end{aligned} \quad (42)$$

C. Evolution of the angles characterizing the spins under radiation reaction

The relative orientation of the momenta can be described by the angles κ_i and γ , see Fig 1. Consequence of $\langle d\mathbf{S}_i/dt \rangle = 0$, the angle γ is conserved:

$$\frac{d}{dt} \cos \gamma \simeq 0 . \quad (43)$$

The angles κ_i were found to evolve due to both the spin-orbit [27] and the spin-spin interactions [29]. Here we compute a third contribution to their evolution, due to magnetic dipole-magnetic dipole interaction:

$$\begin{aligned} \left(\frac{d}{dt} \cos \kappa_i \right)_{DD} &\simeq \frac{1}{\bar{L}} (\hat{\mathbf{S}}_i - \hat{\mathbf{L}} \cos \kappa_i) \cdot \left(\frac{d\mathbf{J}}{dt} \right)_{DD} \\ &= \frac{3Gd_1d_2}{5c^5\mu^2\bar{L}^2r^7} \sin \kappa_i \sum_{j=1}^2 \zeta_{3-j} \left\{ \sum_{k=1}^3 u_k [\sigma_j \cos(k\chi + \delta_i + \delta_j) - \rho_j \sin(k\chi + \delta_i + \delta_j)] \right. \\ &\quad \left. + u_4 \sin \chi [\sigma_j \sin(\delta_i - \delta_j) - \rho_j \cos(\delta_i - \delta_j)] + u_5 [\rho_j \sin(\delta_i - \delta_j) + \sigma_j \cos(\delta_i - \delta_j)] \right\} \end{aligned} \quad (44)$$

with the coefficients u_k

$$\begin{aligned} u_1 &= -u_3 = \mu\bar{A}r(2E\mu r^2 - 3\bar{L}^2) , \\ u_2 &= -2\bar{L}^2(Gm\mu^2r + 3\bar{L}^2) , \\ u_4 &= 2\mu\bar{A}r(2E\mu r^2 - 5\bar{L}^2) , \\ u_5 &= 2\bar{L}^2(4E\mu r^2 + Gm\mu^2r - 5\bar{L}^2) . \end{aligned} \quad (45)$$

Averaging yields the following secular expression for the change of the angle κ_i :

$$\begin{aligned} \left\langle \frac{d\kappa_i}{dt} \right\rangle_{DD} &= \frac{3Gd_1d_2(-2E\mu)^{3/2}}{5c^5\bar{L}^9} \sum_{j=1}^2 \zeta_{3-j} \{ V_1 [\sigma_j \cos(\delta_i + \delta_j) - \rho_j \sin(\delta_i + \delta_j)] \\ &\quad + V_2 [\sigma_j \cos(\delta_i - \delta_j) + \rho_j \sin(\delta_i - \delta_j)] \} , \end{aligned} \quad (46)$$

where $V_{1,2}$ are

$$\begin{aligned} V_1 &= 5\mu\bar{A}^2(4E\bar{L}^2 + 7G^2m^2\mu^3) , \\ V_2 &= 48E^2\bar{L}^4 + 140G^2E\bar{L}^2m^2\mu^3 + 70G^4m^4\mu^6 . \end{aligned} \quad (47)$$

Note that although the detailed expressions of the secular evolutions of E , L and κ_i are different from the corresponding expressions characterizing the gravitational quadrupole-monopole interaction [30], the coefficients a_{1-4} , b_{1-4} , u_{1-5} , $C_{1,2}$, $D_{1,2}$ and $V_{1,2}$ are identical! This is related to the similar structure of the respective Lagrangians.

D. Evolution of the angles characterizing the magnetic moments under radiation reaction

The first order differential equations (31), (39), (43), and (46) together with the algebraic constraints (presented in detail in [27]):

$$S_1 \sin \kappa_1 \cos \psi_1 + S_2 \sin \kappa_2 \cos \psi_2 = 0 , \quad (48)$$

$$\cos \kappa_1 \cos \kappa_2 + \sin \kappa_1 \sin \kappa_2 \cos (\psi_2 - \psi_1) = \cos \gamma . \quad (49)$$

does not form a closed system, due to the presence of the angles λ , α_i , and β_i contained in A_0 , B_0 , and ζ_i . This is a new feature to be contrasted with the spin-orbit, spin-spin and quadrupole-monopole cases. Therefore we need the radiative evolution equations for the above angles either.

For this purpose we remark that λ can be expressed in term of the other enlisted angles as

$$\cos \lambda = (\rho_2 \sigma_1 - \rho_1 \sigma_2) \sin(\delta_1 - \delta_2) + (\rho_1 \rho_2 + \sigma_1 \sigma_2) \cos(\delta_1 - \delta_2) + \zeta_1 \zeta_2 . \quad (50)$$

A second remark is that similarly to $\langle d\mathbf{S}_i/dt \rangle = 0$, we expect $\langle d\mathbf{d}_i/dt \rangle = 0$ to hold. The assumption can be lifted by considering any neutron star model which relates the magnetic dipole moment to other characteristics, like the spin².

Therefore the radiative change of the angles $\alpha_i = \cos^{-1}(\hat{\mathbf{S}}_i \cdot \hat{\mathbf{d}}_i)$ is beyond the order of accuracy of our computation

$$\frac{d}{dt} \cos \alpha_i \simeq 0 , \quad (51)$$

and the only task remains to compute the radiative evolution of

$$\cos \beta_i = \frac{{}^\perp \hat{\mathbf{d}}_i \cdot (\hat{\mathbf{S}}_i \times \hat{\mathbf{L}})}{\sin \kappa_i} , \quad (52)$$

where by

$${}^\perp \hat{\mathbf{d}}_i = \begin{pmatrix} \sin \beta_i \cos \kappa_i \cos \psi_i + \cos \beta_i \sin \psi_i \\ \sin \beta_i \cos \kappa_i \sin \psi_i - \cos \beta_i \cos \psi_i \\ - \sin \beta_i \sin \kappa_i \end{pmatrix} \quad (53)$$

we have denoted the unit vector aligned with the projection of the magnetic dipole moment to the plane perpendicular to the spin of the respective neutron star. The vector product of the projection of the magnetic dipole moment and the direction of the spin, appearing in the subsequent expressions, is readily obtained

$${}^\perp \hat{\mathbf{d}}_i \times \hat{\mathbf{S}}_i = \begin{pmatrix} \sin \beta_i \sin \psi_i - \cos \beta_i \cos \psi_i \cos \kappa_i \\ - \sin \beta_i \cos \psi_i - \cos \beta_i \sin \psi_i \cos \kappa_i \\ \cos \beta_i \sin \kappa_i \end{pmatrix} . \quad (54)$$

Up to terms which do not contribute to the secular expressions

$$\frac{d}{dt} \cos \beta_i \simeq \frac{{}^\perp \hat{\mathbf{d}}_i \times \hat{\mathbf{S}}_i}{L(\chi) \sin \kappa_i} \cdot \frac{d\mathbf{J}}{dt} - \frac{\cos \beta_i}{L(\chi)} \frac{dL}{dt} - \frac{\cos \kappa_i}{\sin \kappa_i} \cos \beta_i \frac{d\kappa_i}{dt} . \quad (55)$$

By virtue of Eq. (35) and Eq. (52) we find that to the leading order in the DD -terms, Eq. (55) becomes:

$$\frac{d}{dt} (\sin \kappa_i \cos \beta_i) \simeq \frac{1}{L} \left\{ {}^\perp \hat{\mathbf{d}}_i \times \hat{\mathbf{S}}_i - \left[({}^\perp \hat{\mathbf{d}}_i \times \hat{\mathbf{S}}_i) \cdot \hat{\mathbf{L}} \right] \hat{\mathbf{L}} \right\} \left(\frac{d\mathbf{J}}{dt} \right)_{DD} . \quad (56)$$

As $(d\mathbf{J}/dt)_N \sim \hat{\mathbf{L}}$ (for which the expression in brackets acts as a projector) we find that similarly to the angles κ_i , the radiative evolution of the angles β_i receives no Newtonian contribution. Employing Eq. (44), the detailed expression

² Actual magnetar models relate the non-radiative spin precession to the magnetic dipole-moment. Concerning the evolution of the magnetic field, they deal only with non-radiative evolution, ranging from allowing no evolution at all [34] to a non-linear magnetic field decay through the Hall-drift [35].

of Eq.(56) turns out to be

$$\begin{aligned} \frac{d}{dt} \cos \beta_i = & -\frac{3Gd_1d_2}{5c^5\mu^2\bar{L}^2r^7} \frac{\sin \beta_i}{\sin \kappa_i} \sum_{j=1}^2 \zeta_{3-j} \left\{ \sum_{k=1}^3 u_k [\rho_j \cos(k\chi + \delta_i + \delta_j) + \sigma_j \sin(k\chi + \delta_i + \delta_j)] \right. \\ & \left. -u_4 \sin \chi [\sigma_j \cos(\delta_i - \delta_j) + \rho_j \sin(\delta_i - \delta_j)] + u_5 [\sigma_j \sin(\delta_i - \delta_j) - \rho_j \cos(\delta_i - \delta_j)] \right\} \end{aligned} \quad (57)$$

and its average over a radial period gives

$$\begin{aligned} \left\langle \frac{d\beta_i}{dt} \right\rangle_{DD} = & -\frac{3Gd_1d_2(-2E\mu)^{3/2}}{5c^5\bar{L}^9\sin \kappa_i} \sum_{j=1}^2 \zeta_{3-j} \left\{ V_1 [\rho_j \cos(\delta_i + \delta_j) + \sigma_j \sin(\delta_i + \delta_j)] \right. \\ & \left. + V_2 [\sigma_j \sin(\delta_i - \delta_j) - \rho_j \cos(\delta_i - \delta_j)] \right\} . \end{aligned} \quad (58)$$

Note that the coefficients appearing in the instantaneous and averaged loss of the angles β_i agree with the coefficients (45) and (47) in the radiative change of κ_i .

IV. CONCLUDING REMARKS

We have given a closed system of differential equations governing the evolution of a set of dynamical and geometrical variables under the magnetic dipole - magnetic dipole contribution to the radiation reaction. These are the energy, magnitude of orbital angular momentum, angles between the spins and orbital angular momentum and angles between the dipole moments and the respective spins. These evolutions add to previously derived PN, 2PN, SO, SS and QM contributions. For sufficiently strong magnetic fields this new contribution is of second PN order.

As a check of the above results we compare the circular orbit limit of the energy loss with the previous result of Ioka and Taniguchi [32]. Imposing the circularity condition in an average sense (as described in Ref. [30]), the following relations hold

$$\bar{E}_N = E - \bar{E}_{DD} = -\frac{Gm\mu}{2r_0} , \quad \bar{L}_N^2 = \bar{L}^2 = Gm\mu^2 r_0 . \quad (59)$$

Here r_0 is the radius of the unperturbed circular orbit and the angular average \bar{E}_{DD} of (7) is

$$\bar{E}_{DD} = \frac{d_1d_2\mathcal{A}_0}{2r_0^3} . \quad (60)$$

Inserting E and \bar{L} from Eqs. (59) into the expression of the energy loss (31) we obtain the radiative loss of energy for the previously defined circular orbit:

$$\left\langle \frac{dE}{dt} \right\rangle = -\frac{32G^4m^3\mu^2}{5c^5r_0^5} \left(1 - \frac{12d_1d_2\mathcal{A}_0}{Gm\mu r_0^2} \right) . \quad (61)$$

Comparison with the corresponding result of [32] is achieved by remarking that Eq. (8) can be written in the form

$$\mathcal{A}_0 = 3(\hat{\mathbf{L}} \cdot \hat{\mathbf{d}}_1)(\hat{\mathbf{L}} \cdot \hat{\mathbf{d}}_2) - \hat{\mathbf{d}}_1 \cdot \hat{\mathbf{d}}_2 , \quad (62)$$

and by computing the averaged radius of the quasi-circular orbit in terms of the radius of the Newtonian circular orbit r_0 :

$$\langle r \rangle = r_0 \left(1 + \frac{3d_1d_2\mathcal{A}_0}{2m\mu r_0^2} \right) . \quad (63)$$

After the required series expansion we find that our energy loss, when specialized to circular orbits and expressed in terms of $\langle r \rangle$, is in perfect agreement with the energy loss contribution due to magnetic dipole - magnetic dipole interaction, given in Eq.(16) of Ref. [32].

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- [1] A. Abramovici et al., *Science* **256**, 325 (1992).
 - [2] C. Bradaschia et al., *Nucl. Instrum. Methods Phys. Res. A* **289**, 518 (1990).
 - [3] J. Hough, in *Proceedings of the Sixth Marcell Grossmann Meeting*, edited by H. Sato and T. Nakamura (World Scientific, Singapore, 1992), p. 192.
 - [4] K. Kuroda et al., in *Proceedings of International Conference on Gravitational Waves: Sources and Detectors*, edited by I. Ciufolini and F. Fidecaro (World Scientific, Singapore, 1997), p. 100.
 - [5] P. Bender, I. Ciufolini, K. Danzmann, W. M. Folkner, J. Hough, D. Robertson, A. Rüdiger, M. C. W. Sandford, R. Schilling, B. F. Schutz, R. Stebbins, T. Sumner, P. Touboul, S. Vitale, H. Ward, and W. Winkler: *LISA: Pre-Phase A Report (MPQ 208)* (Max-Planck Institut für Quantenoptik, Garching, Germany, 1996).
 - [6] B. F. Schutz, in *Proceedings of the 1997 Alpbach Summer School on Fundamental Physics in Space*, edited by A. Wilson, gr-qc/9710079.
 - [7] L. Blanchet, *Post-Newtonian Theory and its Application*, to appear in the Proc. of the 12th Workshop on General Relativity and Gravitation, ed. M. Shibata, gr-qc/0304014 (2003).
 - [8] P.R. Brady, J.D.E. Creighton, and K.S. Thorne *Phys. Rev. D* **58**, 061501 (1998).
 - [9] A. Buonanno, Y. Chen, and M. Vallisneri, *Phys. Rev. D* **67**, 024016 (2003).
 - [10] P. C. Peters, *Phys. Rev.* **136**, B1224 (1964).
 - [11] G. D. Quinlan and S. L. Shapiro, *Astrophys. J.* **321**, 199 (1987).
 - [12] D. Hills and P. L. Bender, *Astrophys. J. Lett.* **445**, L7 (1995).
 - [13] K. Martel and E. Poisson, *Phys. Rev. D* **60**, 124008 (1999).
K. Martel, *Detection of Gravitational Waves from Eccentric Compact Binaries*, gr-qc/9908043.
 - [14] H. Falcke, P. L. Biermann, W. J. Duschl and P. G. Mezger, *Astron. Astrophys.* **270**, 102 (1993).
 - [15] L. Bildsten and C. Cutler, *Astrophys. J.* **400**, 175 (1992).
 - [16] C.S. Kochanek, *Astrophys. J.* **398**, 234 (1992).
 - [17] C. Kouveliotou, et al., *Nature* **393**, 235 (1998).
 - [18] P. C. Peters and J. Mathews, *Phys. Rev.* **131**, 435 (1963).
 - [19] A. Buonanno, Y. Chen and M. Vallisneri, *Phys. Rev. D* **67**, 104025 (2003).
 - [20] P. Grandclément and V. Kalogera, *Phys. Rev. D* **67**, 082002 (2003).
 - [21] A. Gopakumar and B. R. Iyer, *Phys. Rev. D* **56**, 7708 (1997).
 - [22] B. M. Barker and R. F. O'Connell, *Phys. Rev. D* **12**, 329 (1975).
 - [23] T. A. Apostolatos, C. Cutler, G. J. Sussman and K. S. Thorne, *Phys. Rev. D* **49**, 6274 (1994).
 - [24] L. Kidder, C. Will, and A. Wiseman, *Phys. Rev. D* **47**, 4183 (1993).
 - [25] L. Kidder, *Phys. Rev. D* **52**, 821 (1995).
 - [26] R. Rieth and G. Schäfer, *Class. Quantum Grav.* **14**, 2357 (1997).
 - [27] L. Á. Gergely, Z. Perjés, and M. Vasúth, *Phys. Rev. D* **58**, 124001 (1998).
 - [28] L. Á. Gergely, *Phys. Rev. D* **61**, 024035 (2000).
 - [29] L. Á. Gergely, *Phys. Rev. D* **62**, 024007 (2000).
 - [30] L. Á. Gergely and Z. Keresztes, *Phys. Rev. D* **67**, 024020 (2003).
 - [31] E. Poisson, *Phys. Rev. D* **57**, 5287 (1998).
 - [32] K. Ioka and T. Taniguchi, *Astrophys. J.* **537**, 327 (2000).
 - [33] L. Á. Gergely, Z. Perjés, and M. Vasúth, *Astrophys. J. Suppl. Ser.* **126**, 79 (2000).
 - [34] T. Regimbau and J. de Freitas Pacheco, *Astron. and Astrophys.* **374**, 182 (2001).
 - [35] U. Geppert and M. Rheinhardt, astro-ph/0207065, to appear in *Astron. and Astrophys.*